Simulations of the dynamo effect with the lattice Boltzmann method

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Abstract

Simulations of the dynamo effect require the simultaneous integration of the Navier-Stokes equation and of the induction equation of electrodynamics. We present a hybrid method in which the Navier-Stokes equation is solved with a lattice Boltzmann method and the induction equation is treated with a spectral method.

Key words: lattice Boltzmann method, spectral methods, magnetohydrodynamics, dynamo effect

1 Introduction

The "dynamo effect" is the conversion of mechanical energy into magnetic energy. It is widely accepted that planetary magnetic fields are created by a dynamo effect, where the mechanical energy resides in the motion of a liquid core, which is electrically conducting because it consists mostly of iron. The dynamo effect in the earth and other planets has so far been studied mostly numerically, but laboratory experiments have also been built in recent years (for reviews, see refs. [?] and [?]). The combination of both approaches is our most promising tool for gaining insight into planetary dynamos [?].

Both the design and the data analysis of the laboratory experiments require heavy numerical computations [?] [?]. These simulations are complicated by the fact that unlike the real earth, the experiments always contain some mechanical structure of complex geometry. That's why flow simulations accompanying such experiments are best done with a flexible method like the lattice Boltzmann method (LBM). Previous dynamo simulations have mostly used spectral methods. There are three reasons why spectral methods are favored when computing magnetic fields. First, the boundary conditions at a conductor/insulator interface are nonlocal. They acquire simple expressions when the magnetic field is spectrally decomposed in a suitable function base provided that the fluid is confined to plane layers, infinite cylinders, or spheres. Second, the condition that there be no magnetic monopoles can be implemented particularly accurately in a spectral method. Related to this point is the third reason in favor of spectral methods, which is their convergence properties. When dealing with dynamos one is first faced with a binary question: Is a given flow capable of generating a magnetic field, yes or no? Because of this dividing line between dynamos and non-dynamos, small numerical errors can lead to qualitatively wrong results, e.g. a zero magnetic field even though the flow is capable of dynamo action or vice versa. In order to prevent catastrophes of this type, it is better to use an accurate high order scheme to simulate the magnetic field.

The experiments which are running or being planned at the moment have outer boundaries which can be reasonably approximated by one of the geometries amenable to spectral methods. Mechanical structures inside the experimental cell which serve to drive or guide the flow of the liquid conductor are made of material of similar electrical conductivity as the fluid. They thus constrain the flow but not the electrical currents. We are therefore prompted to develop methods which use spectral methods for the equations of electrodynamics and the LBM for solving the Navier-Stokes equation. We demonstrate this coupling strategy here for a case in which a fully spectral method can also be used for validation.

2 The mathematical model

We consider the model [**?**] which has inspired the so called "Karlsruhe dynamo experiment" [**?**]. An infinite expanse of liquid conductor is set into motion such that the velocity field consists of a periodic array of right handed helices. A box with periodic boundary conditions is the appropriate geometry to deal with this problem.

The equations for the non-dimensional velocity $\boldsymbol{u}(\boldsymbol{r},t)$ and magnetic fields, $\boldsymbol{B}(\boldsymbol{r},t)$, are solved in the following form:

$$\partial_t \boldsymbol{B} + \nabla \times (\boldsymbol{B} \times \boldsymbol{u}) = \frac{1}{Rm} \nabla^2 \boldsymbol{B} \quad , \quad \nabla \cdot \boldsymbol{B} = 0 \tag{1}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \tag{2}$$

Two control parameters appear: The Reynolds number Re and the magnetic Reynolds number Rm. p is the pressure. $\boldsymbol{F}(\boldsymbol{r})$ is a forcing which maintains the flow against viscous dissipation and the magnetic force $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$. $\boldsymbol{F}$ is chosen to be

$$\boldsymbol{F} = \frac{2}{Re}(\frac{\pi}{a})^2 \boldsymbol{u}_0 \tag{3}$$

with

$$\boldsymbol{u}_0 = \begin{pmatrix} \sqrt{2}\sin(\frac{2\pi}{a}x)\cos(\frac{2\pi}{a}y) \\ -\sqrt{2}\cos(\frac{2\pi}{a}x)\sin(\frac{2\pi}{a}y) \\ 2\sin(\frac{2\pi}{a}x)\sin(\frac{2\pi}{a}y) \end{pmatrix}. \tag{4}$$

The equations are solved subject to periodic boundary conditions in the domain $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$. As long as $\boldsymbol{B} = 0$, $\boldsymbol{u}_0$ is a steady state solution of (**??**). At large enough Re, this solution becomes unsteady. Any magnetic field generated by the dynamo effect will also distort the velocity field.

The spectral method is analogous to the one described for example in [**?**] and uses a Fourier decomposition of $\boldsymbol{B}$:

$$\boldsymbol{B}(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{B}}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\boldsymbol{r}} \tag{5}$$

where the sum is over all wavenumbers $\boldsymbol{k}$ compatible with the boundary conditions. Equation (**??**) is in spectral space:

$$\frac{d}{dt}\hat{\boldsymbol{B}}_{\boldsymbol{k}} + i\boldsymbol{k} \times (\widehat{\boldsymbol{B} \times \boldsymbol{u}})_{\boldsymbol{k}} = -\frac{1}{Rm}|\boldsymbol{k}|^2 \hat{\boldsymbol{B}}_{\boldsymbol{k}} \tag{6}$$

where $(\widehat{\boldsymbol{B} \times \boldsymbol{u}})_{\boldsymbol{k}}$ is the component with wavenumber $\boldsymbol{k}$ of the spectral decomposition of $\boldsymbol{B} \times \boldsymbol{u}$:

$$\boldsymbol{B} \times \boldsymbol{u} = \sum_{\boldsymbol{k}} (\widehat{\boldsymbol{B} \times \boldsymbol{u}})_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}}. \tag{7}$$

The computation of $\boldsymbol{B} \times \boldsymbol{u}$ in spectral space requires a costly convolution product. For this reason, the product $\boldsymbol{B} \times \boldsymbol{u}$ is obtained by transforming $\boldsymbol{B}$ into direct space, taking $\boldsymbol{u}$ from the LBM, computing the product in direct space, and transforming the result back into spectral space. The transformations are done with an FFT algorithm, which requires on the order of $N^3 \log N$ operations where N^3 is the number of Fourier modes in (**??**). Time marching is done with a second order Adams-Bashforth scheme for the $\boldsymbol{B} \times \boldsymbol{u}$ term coupled to a Crank-Nicolson scheme for the diffusion term. If t_n denotes the times at which the unknowns are computed, the time marching scheme is in formula:

$$\left(1 + \frac{h}{2}\frac{|\boldsymbol{k}|^2}{Rm}\right)\hat{\boldsymbol{B}}_{\boldsymbol{k}}(t_{n+1}) = \left(1 - \frac{h}{2}\frac{|\boldsymbol{k}|^2}{Rm}\right)\hat{\boldsymbol{B}}_{\boldsymbol{k}}(t_n) + \\ i\boldsymbol{k}\frac{h}{2}(-3(\widehat{\boldsymbol{B}\times\boldsymbol{u}})_{\boldsymbol{k}}(t_n) + (\widehat{\boldsymbol{B}\times\boldsymbol{u}})_{\boldsymbol{k}}(t_{n-1})) \tag{8}$$

This time step is second order in time, just as the time step in the LBM below.

The velocity field is updated in parallel with an LBM scheme which uses the BGK approximation for the collision operator and the standard 19 velocity model. In the units used here, this method is described by the following equations for the distribution functions f_i:

$$f_i(\boldsymbol{r} + \boldsymbol{v}_i\Delta t, t + \Delta t) - f_i(\boldsymbol{r}, t) = \\ -\frac{\Delta t}{\tau + \Delta t/2}[f_i(\boldsymbol{r}, t) - f_i^{(0)}(\boldsymbol{r}, t)] \quad , \quad i = 0..18 \tag{9}$$

$$\frac{1}{n}f_i^0 = w_i\left[1 + 3\frac{\boldsymbol{v}_i \cdot \tilde{\boldsymbol{u}}}{c^2} - \frac{3}{2}\left(\frac{\tilde{\boldsymbol{u}}}{c}\right)^2 + \frac{9}{2}\left(\frac{\boldsymbol{v}_i \cdot \tilde{\boldsymbol{u}}}{c^2}\right)^2\right] \tag{10}$$

$$n = \sum_i f_i \quad , \quad \tilde{\boldsymbol{u}} = \sum_i \boldsymbol{v}_i f_i \quad , \quad \boldsymbol{u} = \tilde{\boldsymbol{u}} - -\tau\boldsymbol{F} - \tau(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \tag{11}$$

The sound velocity c is fixed by $c = \Delta x/\Delta t$, where Δx and Δt are the mesh size and the time step, respectively. Because the LBM and the spectral methods need to exchange information, we choose the grids of both methods to be identical. The FFT used in the spectral part of the code is most efficient if $\Delta x = a2^{-N}$ with N some integer. The resolution is chosen such that the Mach number $|\boldsymbol{u}|/c$ never exceeds 0.1. The relaxation parameter τ is determined by $\tau = 3/(Re \cdot c^2)$. n is the particle density. $\boldsymbol{v}_i$ are the particle velocities and w_i the associated weights.

In summary, the LBM and the spectral method are independent of each other except for two data transfers per time step. The LBM passes its $\boldsymbol{u}$ to the spectral method so that $\boldsymbol{B} \times \boldsymbol{u}$ can be computed, and the spectral method computes the magnetic force $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$ and transmits it (in direct space) to the LBM where it is needed in (**??**).

3 Numerical Results

We give here a validation of the method presented in the previous section by comparing it to a purely spectral method in which (**??**) is solved by Fourier decomposition, too [**?**]. For an easy comparison, parameters are chosen such that a stationary state rather than a time dependent state is reached after a sufficiently long integration. The results are listed in table **??**. All runs are for $Re = 1$, $Rm = 30$, $a = 1$ and they have been started from the initial conditions $\boldsymbol{u}(\boldsymbol{r}, t = 0) = 0$ and $\boldsymbol{B}(\boldsymbol{r}, t = 0) = 1/2\hat{\boldsymbol{x}}$ where $\hat{\boldsymbol{x}}$ is the unit vector in $x-$direction. The time evolution has been integrated up to $t = 20$. Table **??** compares kinetic and magnetic energies, E_{kin} and E_B, defined by

$$E_{kin} = \int_0^a dx \int_0^a dy \int_0^a dz \; \boldsymbol{u}^2/2 \quad \text{and} \quad E_B = \int_0^a dx \int_0^a dy \int_0^a dz \; \boldsymbol{B}^2/2. \tag{12}$$

The energies are computed with the trapezoidal rule in all codes, even though an exact integration is possible in the spectral method. For stability reasons, the time step had to be chosen smaller for the LBM than for the spectral method. At higher Re, the stability limits for both methods become identical. The execution time per time step is larger for the hybrid code by roughly a factor of 2. As expected, the results in table **??** become independent of spatial resolution already at lower resolution for the purely spectral method than for the hybrid code. This is a manifestation of the better convergence properties of the spectral method and of the fact that this particular problem is well suited for Fourier expansions. At large resolutions, both methods yield results in reasonable agreement which validates the hybrid code.

4 Conclusion

It has been shown that it is possible to couple an LBM with a spectral method. In the test case presented here, both the hybrid method and a fully spectral method can be used to solve the problem. The purely spectral method turns out to be more efficient in this case. The hybrid method on the other hand offers the possibility to include mechanical walls and to perform simulations of dynamo experiments which are more realistic than those done up to now.

method	resolution	Δt	E_{kin}	E_B
S	8^3	2×10^{-3}	0.7737	2.1568
		1×10^{-3}	0.7737	2.1568
	16^3	2×10^{-3}	0.8482	1.4271
		1×10^{-3}	0.8482	1.4271
	32^3	2×10^{-3}	0.8482	1.4268
LBM	8^3	1×10^{-3}	0.6805	1.586
		2.5×10^{-4}	0.668	1.598
	16^3	2.5×10^{-4}	0.819	1.312
	32^3	1×10^{-4}	0.842	1.401

Table 1
Results of comparative runs for the purely spectral method (S) and the the hybrid code in which the Navier-Stokes equation is solved with the lattice Boltzmann method (LBM).

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